CP502 Advanced Fluid Mechanics
Problem Set #2 in Compressible Fluid Flow (Jan - May 2012)

Steady, quasi one-dimesional, isentropic, compressible flow of an ideal gas in a variable area duct

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(1) Consider the following equations governing the steady, isentropic (adiabatic and inviscid), quasi one-dimensional, compressible flow:

$$d(A\rho u) = 0 \tag{2.1}$$
$$dh + u\,du = 0 \tag{2.2}$$
$$dp + \rho u\,du = 0 \tag{2.3}$$

If the given flow experiences negligible changes in its density (that is, if the flow is assumed to be incompressible) then show that the given flow is described by the following equations:

$$A\,u = \text{constant}; \quad h + \frac{u^2}{2} = \text{constant}; \quad \frac{p}{\rho} + \frac{u^2}{2} = \text{constant}$$

Hence, show that a diverging duct compresses, heats and slows down a steady, inviscid, incompressible flow through it.

And, show that a converging duct expands, cools and speeds up a steady, inviscid, incompressible flow through it.

(2) Using the momentum balance (Equation (2.3)) describing the steady, inviscid, quasi one-dimensional flow, explain that (i) the velocity increase of such a flow is always accompanied by the pressure decrease and (ii) the velocity decrease is always accompanied by the pressure increase. *Note that the above nature of a flow is common for both compressible and incompressible flows.*

(3) Show that the steady, isentropic, quasi one-dimensional, compressible flow through a varying area duct shall be governed by the follwing equation:

$$\frac{du}{dA} = \frac{u}{A\,(M^2 - 1)} \tag{2.4}$$

(a) For a compressible flow at subsonic speeds, show that decreasing duct area increases the flow speed and that a increasing duct area decreases the flow speed.

(b) For a compressible flow at supersonic speeds, show that decreasing duct area decreases the flow speed and that a increasing duct area increases the flow speed.

(4) Show that, at $M = 1$, du/u can be finite only if the area of the duct is at its minimum. That is, the sonic speed can be attained only at the throat of the duct. *Note: It is, however, not necessary for M to be always 1 at the throat. If M is not 1 at the throat then the velocity attains a maximum or minimum there, depending on whether the flow is subsonic or supersonic.*

(5) You may work out this problem if you are curious.

(a) Show that the increase/decrease in the flow speed in the case of a compressible flow is relatively greater than that for the incompressible flow.

(b) We have shown in Problem (3), a duct with increasing area is needed to increase the flow speed for a compressible flow at supersonic speeds. Show that the reason for this remarkable behaviour is that at supersonic speeds the density decreases faster than the velocity increases and therefore the area must increase to maintain continuity of mass.

Hint: Using the definition for speed of sound in the momentum equation describing the steady, isentropic, quasi one-dimensional, compressible flow, show that

$$\frac{d\rho}{\rho} = -M^2 \, \frac{du}{u} \, .$$

and that for cases in which M is a nonzero constant, $\rho\, u \propto u^{(1-M^2)}$. Use this expression to show that, for the case of $M > 1$, the decrease in density is greater than the increase in velocity.

(6) Show that the steady, one-dimensional, isentropic, compressible flow of an ideal gas with constant specific heats can be described by the following equations:

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} \tag{2.5}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2 \tag{2.6}$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{2.7}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{1}{\gamma-1}} \tag{2.8}$$

where p_0, T_0 and ρ_0 are the stagnation (where fluid is assumed to be at rest) properties, p, T and ρ are the properties at Mach number M and γ is the specific heat ratio (assumed to be a constant).

(7) Show that the mass flow rate in a steady, one-dimensional, isentropic, compressible flow of an ideal gas with constant specific heats is given by the following equations:

$$\dot{m} = A\, M\, p \sqrt{\frac{\gamma}{R\,T}} \tag{2.9}$$

$$\dot{m} = A\, M\, p_0 \sqrt{\frac{\gamma}{R\,T_0}} \left(\frac{1}{1 + \frac{\gamma-1}{2} M^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{2.10}$$

(8) A large air reservoir contains air at a temperature of 400 K and a pressure of 600 kPa. The air reservoir is connected to a second chamber through a converging duct whose exit area is 100 mm^2. The pressure inside the second chamber can be regulated independently. Assuming steady, isentropic flow in the duct, calculate the exit Mach number, exit temperature, and mass flow rate through the duct when the pressure in the second chamber is (i) 600 kPa, (ii) 500 kPa, (iii) 400 kPa, (iv) 300 kPa and (v) 200 kPa.

(9) *This problem is from page 891 of Çengel, Y.A. & Boles (3rd edition):* Air at 900 kPa and 400 K enters a converging nozzle with a negligible velocity. The throat area of the nozzle is 10 cm^2. Assuming isentropic flow, calculate and plot the exit pressure, the exit mach number, the exit velocity, and the mass flow rate versus the back pressure P_b for 900 $\leq P_b \leq$ 100 MPa.

(10) Consider a converging-diverging duct with a circular cross-section for a mass flow rate of 3 kg/s of air and inlet stagnation conditions of 1400 kPa and 200^{o}C. Assume that the flow is isentropic and the exit pressure is 100 kPa. Plot the pressure and temperature of the air flow along the duct as a function of M. Plot also the diameter of the duct as a function of M.

(11) Air at approximately zero velocity enters a converging-diverging duct at a stagnation pressure and a stagnation temperature of 1000 kPa and 480 K, respectively. Throat area of the duct is 0.002 m^2. The flow inside the duct is isentropic, and the exit pressure is 31.7 kPa. For air, γ = 1.4 and R = 287 J/kg. Determine (i) the exit Mach number, (ii) the exit temperature, (iii) the exit area of the duct, and (iii) the mass flow rate through the duct.