CP303 — Set #0 worked out (January-May 2012)

Solutions to Differential Equations in Reaction Engineering

---

(1) Solve the following differential equations for the given initial conditions:

(i) $dx/dt = a\,x$ with $x = x_o$ at $t = 0$.

Rearranging the above to integrate we get

$$\int \frac{dx}{x} = \int a\,dt$$

which gives

$$\ln x = a\,t + const$$

Since $x = x_o$ at $t = 0$, we get $const = ln x_o$. Therefore,

$$\ln x = a\,t + \ln x_o$$

which gives

$$\ln\left(\frac{x}{x_o}\right) = at$$

and thus

$$x = x_o \exp(at)$$

(ii) $dx/dt = a\,(1 + \epsilon x)$ with $x = 0$ at $t = 0$.

Rearranging the above to integrate we get

$$\int \frac{dx}{1 + \epsilon x} = \int a\,dt$$

which gives

$$\frac{\ln(1 + \epsilon x)}{\epsilon} = a\,t + const$$

Since $x = 0$ at $t = 0$, we get $const = ln(1)/\epsilon = 0$. Therefore,

$$\frac{\ln(1 + \epsilon x)}{\epsilon} = a\,t$$

which gives

$$\ln(1 + \epsilon x) = a\,\epsilon\,t$$

and thus

$$x = \frac{\exp(a\,\epsilon\,t) - 1}{\epsilon}$$

(2) Solve the following differential equations for a non-zero value of ϵ and give the solutions in terms of integrating constants:

(i) $dx/dt = a\,[(1-x)/(1+\epsilon x)]$.

Rearranging the above to integrate we get

$$\int \frac{1+\epsilon x}{1-x}\,dx = \int a\,dt$$

Substituting $y = 1-x$ gives $x = 1-y$ and $dx = -dy$, and hence the above expression becomes

$$\int -\frac{1+\epsilon(1-y)}{y}\,dy = \int a\,dt$$

which is rearranged to give

$$\int \left(-\frac{1+\epsilon}{y} + \epsilon\right)\,dy = \int a\,dt$$

Upon integration of the above we get

$$-(1+\epsilon)\,\ln y + \epsilon\,y = a\,t + const$$

Replacing the y in the above expression by $(1-x)$, we get

$$-(1+\epsilon)\,\ln(1-x) + \epsilon\,(1-x) = a\,t + const$$

(ii) $dx/dt = a\,[(1-x)^2/(1+\epsilon x)]$.

Rearranging the above to integrate we get

$$\int \frac{1+\epsilon x}{(1-x)^2}\,dx = \int a\,dt$$

Substituting $y = 1-x$ gives $x = 1-y$ and $dx = -dy$, and hence the above expression becomes

$$\int -\frac{1+\epsilon(1-y)}{y^2}\,dy = \int a\,dt$$

which is rearranged to give

$$\int \left(-\frac{1+\epsilon}{y^2} + \frac{\epsilon}{y}\right)\,dy = \int a\,dt$$

Upon integration of the above we get

$$\frac{1+\epsilon}{y} + \epsilon\,\ln y = a\,t + const$$

Replacing the y in the above expression by $(1-x)$, we get

$$\frac{1+\epsilon}{1-x} + \epsilon\,\ln(1-x) = a\,t + const$$

(iii) $dx/dt = a\,[(1-x)^2/(1+\epsilon x)^2]$.

Rearranging the above to integrate we get

$$\int \frac{(1+\epsilon x)^2}{(1-x)^2}\,dx = \int a\,dt$$

Substituting $y = 1-x$ gives $x = 1-y$ and $dx = -dy$, and hence the above expression becomes

$$\int -\frac{(1+\epsilon-\epsilon\,y)^2}{y^2}\,dy = \int a\,dt$$

which is rearranged to give

$$\int -\left(\frac{(1+\epsilon)^2}{y^2} - \frac{2\,\epsilon\,(1+\epsilon)}{y} + \epsilon^2\right)\,dy = \int a\,dt$$

Upon integration of the above we get

$$\frac{(1+\epsilon)^2}{y} + 2\,\epsilon\,(1+\epsilon)\,\ln y - \epsilon^2\,y = a\,t + const$$

Replacing the y in the above expression by $(1-x)$, we get

$$\frac{(1+\epsilon)^2}{1-x} + 2\,\epsilon\,(1+\epsilon)\,\ln(1-x) - \epsilon^2\,(1-x) = a\,t + const$$

(3) Solve $dx/dt = a\,(x-p)\,(x-q)$ and give the solutions in terms of integrating constants.

(i) For $p \neq q$

Rearranging the above to integrate we get

$$\int \frac{dx}{(x-p)\,(x-q)} = \int a\,dt$$

which can be written as

$$\int \left(\frac{1}{p-q}\right)\left(\frac{1}{x-p} - \frac{1}{x-q}\right)\,dx = \int a\,dt$$

Upon integration we get

$$\left(\frac{1}{p-q}\right)(\ln(x-p) - \ln(x-q)) = a\,t + const$$

which is simply

$$\left(\frac{1}{p-q}\right)\ln\left(\frac{x-p}{x-q}\right) = a\,t + const$$

(ii) For $p = q$

Rearranging the above to integrate we get

$$\int \frac{dx}{(x-p)^2} = \int a\,dt$$

Upon integration we get

$$\frac{-1}{x-p} = a\,t + const$$

(4) Solve $dx/dt = (c + gx)/(a + bx)$ for the initial condition $x = 0$ at $t = 0$.

Rearranging the above to integrate we get

$$\int \frac{a + bx}{c + gx}\, dx = \int dt$$

Substituting $y = c + gx$ gives $x = (y - c)/g$ and $dx = dy/g$, and hence the above expression becomes

$$\int \frac{a + b(y - c)/g}{g\, y}\, dy = \int dt$$

which is rearranged to give

$$\int \left(\frac{ag - bc}{g^2\, y} + \frac{b}{g^2} \right) dy = \int dt$$

Upon integration of the above we get

$$\frac{ag - bc}{g^2}\, \ln y + \frac{b}{g^2}\, y = t + const$$

Replacing the y in the above expression by $(c + gx)$, we get

$$\frac{ag - bc}{g^2}\, \ln{(c + gx)} + \frac{b}{g^2}\, (c + gx) = t + const$$

Substituting the initial condition $x = 0$ at $t = 0$ in the above, we get

$$\frac{ag - bc}{g^2}\, \ln c + \frac{b}{g^2}\, c = const$$

Removing the $const$ from the above two expressions by subtraction, we get

$$\frac{ag - bc}{g^2}\, \ln \left(\frac{c + gx}{c} \right) + \frac{b}{g}\, x = t$$

(5) Solve the following set of differential equations

$$\begin{aligned} \frac{dC_A}{dt} &= -k_1\, C_A \\ \frac{dC_B}{dt} &= -k_2\, C_B + k_1\, C_A \end{aligned}$$

with the initial condition of $C_A = C_{Ao}$ and $C_B = 0$ at $t = 0$.

(i) For $k_1 \neq k_2$

From the solution to part (i) of Problem (1) (see page 3), we could see that integrating the first differential equation given above,

$$\frac{dC_A}{dt} = -k_1\, C_A \tag{0.1}$$

with the initial condition of $C_A = C_{Ao}$ at $t = 0$, gives

$$C_A = C_{Ao} \exp(-k_1 t) \tag{0.2}$$

Integrating the second differential equation,

$$\frac{dC_B}{dt} = -k_2 C_B + k_1 C_A \tag{0.3}$$

is of course not straight forward since the above differential equation has three variables, which are C_A, C_B and t. Substituting (0.2) in (0.3), we get

$$\frac{dC_B}{dt} = -k_2 C_B + k_1 C_{Ao} \exp(-k_1 t) \tag{0.4}$$

in which C_B and t could not be separated. Therefore, we use the Integrating Factor method to integrate (0.4) as follows:

Rewrite (0.4) as

$$\frac{dC_B}{dt} + k_2 C_B = k_1 C_{Ao} \exp(-k_1 t)$$

Multiply the above equation by the Integrating Factor of the above equation, which is $\exp(k_2 t)$. We then get

$$\exp(k_2 t) \frac{dC_B}{dt} + \exp(k_2 t) k_2 C_B = \exp(k_2 t) k_1 C_{Ao} \exp(-k_1 t)$$

The left-hand-side of the above equation can be combined to give the following,

$$\frac{d}{dt} [\exp(k_2 t) C_B] = \exp(k_2 t) k_1 C_{Ao} \exp(-k_1 t)$$

which could be rewritten as

$$d [\exp(k_2 t) C_B] = k_1 C_{Ao} \exp[(-k_1 + k_2) t] \, dt \tag{0.5}$$

Upon integration (0.5) gives

$$\exp(k_2 t) C_B = \frac{k_1}{-k_1 + k_2} C_{Ao} \exp[(-k_1 + k_2) t] + const \tag{0.6}$$

Substituting the initial condition $C_B = 0$ at $t = 0$ in (0.6), we get

$$0 = \frac{k_1}{-k_1 + k_2} C_{Ao} + const \tag{0.7}$$

Subtracting (0.7) from (0.6), we get

$$\exp(k_2 t) C_B = \frac{k_1}{-k_1 + k_2} C_{Ao} \exp[(-k_1 + k_2) t] - \frac{k_1}{-k_1 + k_2} C_{Ao}$$

which could be rearranged to give

$$C_B = \frac{k_1}{-k_1 + k_2} C_{Ao} [\exp(-k_1 t) - \exp(-k_2 t)]$$

(ii) For $k_1 = k_2$

If $k_1 = k_2$, then (0.5) could not be integrated to get (0.6). We must first substitute $k_1 = k_2$ in (0.5). We then get

$$d\left[\exp\left(k_1\, t\right) C_B\right] = k_1\, C_{Ao}\, dt \tag{0.8}$$

Upon integration (0.8) gives

$$\exp\left(k_1\, t\right) C_B = k_1\, C_{Ao}\, t + const \tag{0.9}$$

Substituting the initial condition $C_B = 0$ at $t = 0$ in (0.9), we get

$$0 = const$$

Therefore, (0.9) gives

$$C_B = k_1\, C_{Ao}\, t\, \exp\left(-k_1\, t\right)$$